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Abstract: If Ω is a bounded domain in $\mathbb{R}^n$, $1 \leq q < p \leq \infty$ and $s = 0, 1, 2, \ldots$, then we clearly have $W^{s,p}(\Omega) \subset W^{s,q}(\Omega)$. We prove that this property does not hold when s in not an integer.

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Abstract - If Ω is a bounded domain in $\mathbb{R}^n$, $1 \le q < p \le \infty$ and $s = 0, 1, 2, \ldots$, then we clearly have $W^{s,p}(\Omega) \subset W^{s,q}(\Omega)$. We prove that this property does not hold when s in not an integer.

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1 A non embedding

In connection with his work on distributional Jacobians [3], H. Brezis asked us whether

$$\text{the inclusion } W^{1/2,3}((0,1)) \subset W^{1/2,2}((0,1)) \text{ holds.} \tag{1}$$

The answer is *negative*. This is counterintuitive at first sight, since $L^3((0,1)) \subset L^2((0,1))$ and $W^{1,3}((0,1)) \subset W^{1,2}((0,1))$; thus, by "1/2 interpolation", we would expect (1) to hold.
Below we shall formulate our main result in a little bit greater generality. The class of *fractional* Sobolev spaces we have in mind is defined as follows. Let Ω be a nontrivial open subset of $\mathbb{R}^n$. Let $1 \le p \le \infty$. With $s = m + \sigma$, $m \in \mathbb{N}_0$ (the natural numbers including 0), and $0 < \sigma < 1$, the fractional Sobolev space $W^{s,p}(\Omega)$ is the collection of all $f \in L^p(\Omega)$ such that its distributional derivatives $D^\alpha f$, $\alpha \le m$, are regular and

$$\max_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+\sigma p}} \, dx \, dy < \infty \, . \tag{2}$$

In this note, we give several proofs of the following

Theorem 1.1 *Let $s > 0$ be a non integer, and let $1 \le q < p \le \infty$. Then there exists some in Ω compactly supported function f such that $f \in W^{s,p}(\Omega)$ but $f \notin W^{s,q}(\Omega)$.*

The same result was obtained independently by J. Van Schaftingen [12], using a proof similar to our second one.

Below we shall discuss three examples, all having their own advantages and disadvantages. In two examples we shall work with a periodic background, in the remaining with a non-periodic one. In the first example we shall work with the Gagliardo semi-norm itself (see (2)). In the other cases our computations will rely on norm equivalences whose proofs are sometimes delicate.

2 The first example

We shall work with the Gagliardo semi-norm. In some sense the first example is elementary.

Before proceeding, let us note that it suffices to establish the following fact: with s, p, q as above and with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ the standard torus,

$$\text{there exists some } g \in W^{s,p}(\mathbb{T}) \setminus W^{s,q}(\mathbb{T}). \tag{3}$$

Proof of "(3) implies Theorem 1.1". Let $g \in W^{s,p}(\mathbb{T}) \setminus W^{s,q}(\mathbb{T})$. Using a partition of unity on $\mathbb{T}$, we find that for some $\varphi \in C^\infty$ supported in some interval of length $< 2\pi$, the function $h := \varphi f$ is in $W^{s,p}(\mathbb{T}) \setminus W^{s,q}(\mathbb{T})$. By the choice of φ, h can be identified with a compactly supported function in $W^{s,p}(\mathbb{R}) \setminus W^{s,q}(\mathbb{R})$.

Consider next some function $\psi \in C_c^\infty(\mathbb{R}^{n-1})$, $\psi \not\equiv 0$. Then clearly $f := \psi \otimes h$ is compactly supported, and belongs to $W^{s,p}(\mathbb{R}^n) \setminus W^{s,q}(\mathbb{R}^n)$.

For all $\lambda > 0$ and all $x_0 \in \mathbb{R}^n$, the mapping $f \mapsto f(\lambda(\,\cdot\, - x_0))$ leaves the space $W^{s,p}(\mathbb{R}^n)$ invariant. Applying this argument our construction yields a function supported in a ball whose radius and centre are at our disposal. $\square$

For $s = m + \sigma$, $m \in \mathbb{N}_0$ and $0 < \sigma < 1$, the periodic fractional order Sobolev space $W^{s,p}(\mathbb{T})$ can be normed with

$$\|f\|_{W^{s,p}(\mathbb{T})} := \|f\|_{L^p} + \left(\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|\Delta_h f^{(m)}(x)|^p}{|h|^{\sigma p+1}}\, dh dx \right)^{1/p} \tag{4}$$

(obvious modification when $p = \infty$). Here, $\Delta_h g(x) := g(x+h) - g(x)$.

We will rely on the Brezis-Lieb lemma [2] that we recall here: if $1 \le p < \infty$, $f_\ell \to f$ a.e. and $\|f_\ell\|_{L^p} \le C$, then

$$\|f_\ell\|_{L^p}^p = \|f\|_{L^p}^p + \|f_\ell - f\|_{L^p}^p + o(1) \text{ as } \ell \to \infty.$$

We also rely on the following straightforward

Lemma 2.1 *We have*

$$\|x \mapsto e^{\imath \ell x}\|_{W^{s,p}} \sim \ell^s \text{ as } \ell \to \infty. \tag{5}$$

Proof. The case $p = \infty$ being left to the reader, we assume that $1 \leq p < \infty$. Clearly, it is enough to consider $0 < s < 1$. Set $f_\ell(x) = e^{\imath \ell x}$. Since $\|f_\ell\|_{L^p} \sim 1$, in order to prove the lemma it suffices to prove that

$$I_\ell := \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|\Delta_h f_\ell(x)|^p}{|h|^{sp+1}} \, dh dx \sim \ell^{sp}.$$

This follows from the identity

$$I_\ell = 2\pi \, \ell^{sp} \int_0^{2\ell\pi} \frac{|e^{\imath\xi} - 1|^p}{|\xi|^{sp+1}} \, d\xi \tag{6}$$

and the fact that the integral in (6) has a positive finite limit as $\ell \to \infty$. □

First proof of Theorem 1.1. We let to the reader the case where $p = \infty$, which is obtained by a rather straightforward modification of the argument below. We thus assume that $p < \infty$.

We will construct by induction on j sequences λ_j and ℓ_j such that

$$x \mapsto g(x) := \sum_{j \geq 1} \lambda_j e^{\imath \ell_j x} \text{ belongs to } W^{s,p} \text{ but not to } W^{s,q}. \tag{7}$$

We pick $\lambda_1 = 1$, $\ell_1 = 1$. Assuming $\lambda_1, \ldots, \lambda_j$, $\ell_1, \ldots, \ell_j$ already constructed, let

$$f_\ell(x) := \frac{1}{j^{1/q} \, \ell^s} e^{\imath \ell x}.$$

By Lemma 5, we have

$$\|f_\ell\|_{W^{s,r}} \sim \frac{1}{j^{1/q}}, \ \forall\, 1 \leq r < \infty.$$

On the other hand, if we write $s = m + \sigma$ then we have $f_\ell \to 0$ and $f_\ell^{(m)} \to 0$ pointwise as $\ell \to \infty$. By the Brezis-Lieb lemma, for $1 \leq r < \infty$ we have, as $\ell \to \infty$,

$$\left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{\imath \ell_k x} + f_\ell(x) \right\|_{W^{s,r}}^r = \left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{\imath \ell_k x} \right\|_{W^{s,r}}^r + \|f_\ell\|_{W^{s,r}}^r + o(1).$$

Thus, for large ℓ, we have

$$\begin{aligned} \left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{\imath \ell_k x} + f_\ell(x) \right\|_{W^{s,p}}^p \leq & \left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{\imath \ell_k x} \right\|_{W^{s,p}}^p \\ & + \frac{K_1}{j^{p/q}} \end{aligned} \tag{8}$$

and

$$\left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{\imath \ell_k x} + f_\ell(x) \right\|_{W^{s,q}}^{q} \geq \left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{\imath \ell_k x} \right\|_{W^{s,q}}^{q} + \frac{K_2}{j}. \tag{9}$$

Using (8) and (9), we construct λ_j and ℓ_j such that

$$\|g\|_{W^{s,p}}^{p} \leq C_p + K_1 \sum_{j \geq 2} \frac{1}{j^{p/q}}$$

and

$$\|g\|_{W^{s,q}}^{q} \geq K_2 \sum_{j \geq 2} \frac{1}{j},$$

and thus g satisfies (7). □

3 The second example

We shall work with lacunary series and Fourier-analytical characterizations of $W^{s,p}(\mathbb{T})$.

Therefore we recall the following characterization of $W^{s,p}(\mathbb{T})$ in terms of Fourier series, see [6, Theorem 3.5.3]. If $f(x) = \sum f_\ell e^{\imath \ell x}$, set

$$f^0 = f_0, \; f^j(x) = \sum_{2^{j-1} < |\ell| \leq 2^j} f_\ell e^{\imath \ell x}, \; \forall\, j \geq 1.$$

If $1 < p < \infty$, then

$$\|f\|_{W^{s,p}(\mathbb{T})} \sim \left(\sum_{j \geq 0} 2^{sjp} \|f^j\|_{L^p}^{p} \right)^{1/p}. \tag{10}$$

To incorporate the extremal cases $p = 1$ and $p = \infty$ we need the following a little bit more technical modification. Let ψ be an infinitely differentiable compactly supported function such that $\psi(x) = 1$ if $|x| \leq 1$. We define

$$\varphi_0(x) := \psi(x), \qquad \varphi_j(x) := \psi(2^{-j}x) - \psi(2^{-j+1}x), \qquad j = 1, 2, \ldots .$$

This results in a smooth dyadic decomposition of unity, i.e.,

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}.$$

If we assume in addition $\operatorname{supp}\psi \subset [-2,2]$, then $\varphi_j(2^j) = 1$ and

$$\operatorname{supp}\varphi_j \subset [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}], \qquad j = 1, 2, \ldots,$$

follow. Just from the Fourier-analytic definition used in [6, Chapt. 3] we derive

$$\|f\|_{W^{s,p}(\mathbb{T})} \sim \left(\sum_{j\geq 0} 2^{sjp} \|\widetilde{f}^j\|^p_{L^p}\right)^{1/p}, \tag{11}$$

where

$$\widetilde{f}^j(x) = \sum_{\ell=-\infty}^{\infty} f_\ell\, \varphi_j(\ell)\, e^{\imath \ell x}, \qquad j = 0, 1, \ldots,$$

and (11) holds for all $p \in [1, \infty]$.

Second proof of Theorem 1.1. We choose

$$\lambda_j := \frac{1}{2^{sj}\, j^{1/q}}, \ \forall\, j \geq 1,$$

and put

$$g(x) := \sum_{j\geq 1} \lambda_j\, e^{\imath 2^j x}.$$

Using either (10) (if $1 < p < \infty$) or (11) (for $p = 1$ or $p = \infty$), we clearly have $g \in W^{s,p}(\mathbb{T}) \setminus W^{s,q}(\mathbb{T})$. $\square$

Note that the above yields an explicit version of our first example, in the sense that the λ_j's and the ℓ_j's are given by explicit formulas.

4 The third example

In this example we apply wavelets. We follow [11, Section 1.7], but see also Meyer [5].

In this perspective, it will be more convenient to construct some

$$g \text{ such that } g \in W^{s,p}_c(\mathbb{R}) \text{ but } g \notin W^{s,q}_c(\mathbb{R}), \tag{12}$$

i.e., we work in the non-periodic context from the very beginning.

Let $k > s+2$ be an integer, and consider father and mother Daubechies wavelets ψ_F and ψ_M, compactly supported and of class C^k. Let, for $j \in \mathbb{N}$ and $m \in \mathbb{Z}$,

$$\psi^j_m(x) = \begin{cases} \psi_F(x-m), & \text{if } j = 0 \text{ and } m \in \mathbb{Z} \\ 2^{(j-1)/2}\, \psi_M(2^{j-1}x - m), & \text{if } j \geq 1 \text{ and } m \in \mathbb{Z} \end{cases}.$$

Set (assuming say $g \in L^1_{loc}$)

$$g_j^m = \int_{\mathbb{R}} \psi_j^m(x)\, g(x)\, dx.$$

Then

$$\|g\|_{W^{s,p}} \sim \left(\sum_{j=0}^{\infty} 2^{j(sp+p/2-1)} \sum_{m\in\mathbb{Z}} |g_j^m|^p \right)^{1/p}, \tag{13}$$

with the obvious modification when $p = \infty$.

Third proof of Theorem 1.1. The generators of the wavelet basis are compactly supported. Without loss of generality we may assume

$$\operatorname{supp} \psi_M \subset [0, N] \tag{14}$$

for some $N = N(s)$ sufficiently large. We put

$$\lambda_j := \frac{1}{2^{j(s+1/2)}\, j^{1/q}}, \qquad j = 1, 2, \ldots \tag{15}$$

Define

$$g := \sum_{j=1}^{\infty} \lambda_j \sum_{m=0}^{2^j-1} \psi_j^m. \tag{16}$$

By (15) and the fact that the ψ_j^m's define an orthonormal basis in $L^2(\mathbb{R})$, we find that $g \in L^2(\mathbb{R})$, and in particular we have

$$g_j^m = \begin{cases} \lambda_j, & \text{if } j \geq 1 \text{ and } 0 \leq m \leq 2^j - 1 \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

By (14) and (16) we have $\operatorname{supp} g \subset [0, N+1]$. Finally, by (13), (15) and (17) we find that g satisfies (12). $\square$

5 Besov spaces and the interpolation argument

Unlike the first proof, the second and the third one are suited to the scale of Besov or Triebel-Lizorkin spaces. This goes beyond the scope of this note. However, we would like to mention that in Example 2 and 3 we already used the identification of our fractional Sobolev spaces as special cases of Besov spaces. More exactly

$$W^{s,p}(\mathbb{T}) = B^s_{p,p}(\mathbb{T}) \qquad \text{and} \qquad W^{s,p}(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n)\,,$$

$s > 0, s \notin \mathbb{N}$, $1 \le p \le \infty$, see [6, 3.5.4] and [10, 2.5.12].
In the framework of Besov spaces a straightforward adaptation of the second proof lead to the following improvement of (3):

$$W^{s,p}(\mathbb{T}) \not\subset B^s_{q,r}(\mathbb{T}) \qquad \text{if} \qquad p \ge q \quad \text{and} \quad r < p\,. \tag{18}$$

Completely analoguous, Example 3 yields the following counterpart for non-periodic spaces

$$W^{s,p}(\Omega) \not\subset B^s_{q,r}(\Omega) \qquad \text{if} \qquad p \ge q \quad \text{and} \quad r < p\,. \tag{19}$$

Here the Besov space on the domain Ω is defined by restriction, i.e., $f \in L^q(\Omega)$ belongs to $B^s_{q,r}(\Omega)$ if there exists some $g \in B^s_{q,r}(\mathbb{R}^n)$ such that

$$f = g \qquad \text{on} \qquad \Omega\,.$$

Some comments to the literature. Necessary and sufficient conditions for embeddings of one Besov space into another can be found in Taibleson [8], S., Triebel [7] and Haroske, Skrzypczak [4]. Whereas in [7] the authors were dealing with the situation on $\mathbb{R}^n$, Taibleson [8] also considered the periodic case. E.g., (18) can be found in [8, Thm. 19(b)]. For smooth domains Ω Haroske and Skrzypczak [4] have proved (19) in the much more general context of Besov-Morrey spaces.

Finally, for convenience of the reader, we will comment on the "interpolation argument" from page 1. We restrict ourselves to real and complex interpolation. It is known that

$$(L^u(0,1), W^{1,u}(0,1))_{1/2,r} = B^{1/2}_{u,r}(0,1)\,, \qquad 1 \le r \le \infty\,.$$

Now, choosing $u = r = 3$ we conclude

$$\begin{aligned} W^{1/2,3}(0,1) &= (L^3(0,1), W^{1,3}(0,1))_{1/2,3} \\ &\hookrightarrow (L^2(0,1), W^{1,2}(0,1))_{1/2,3} = B^{1/2}_{2,3}(0,1)\,. \end{aligned}$$

The Besov space $B^{1/2}_{2,3}(0,1)$ does not belong to the scale of fractional Sobolev spaces under consideration, it is just a space containing $W^{1/2,2}(0,1) = B^{1/2}_{2,2}(0,1)$. Similarly for the complex method we obtain that

$$[L^u(0,1), W^{1,u}(0,1)]_{1/2} = F^{1/2}_{u,2}(0,1)\,, \qquad 1 < u < \infty\,.$$

Here $F^{1/2}_{u,2}(0,1)$ denotes a Lizorkin-Triebel space. Again choosing $u = 3$ we conclude

$$\begin{aligned} F^{1/2}_{3,2}(0,1) &= [L^3(0,1), W^{1,3}(0,1)]_{1/2} \\ &\hookrightarrow [L^2(0,1), W^{1,2}(0,1)]_{1/2} = W^{1/2,2}(0,1)\,. \end{aligned}$$

The Lizorkin-Triebel space $F^{1/2}_{3,2}(0,1)$ does also not belong to the scale of fractional Sobolev spaces, it is just a space embedded into $W^{1/2,2}(0,1)$. For all this we refer to [1, 6.4] and [9, 2.4].

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